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The Lagrangian Picture of Fluid Motion

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The lagrangian picture of fluid motion

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This paper is introductory to the Theme Issue of *Philosophical Transactions*, which is devoted to the lagrangian description of fluid motions. In much of the literature of the last few decades attention has been focused especially on the eulerian description, in which the time-dependent flow is observed at a given position. This is an important aspect of many experiments, in which velocity and pressure fields are measured at some point. On the other hand, many flow visualizations direct the attention of the observer to the motions of individual particles, and to the phenomena associated with such observations. The lagrangian picture is appropriate for such experiments; new and different phenomena can be seen and understood in a rather simpler manner than in the eulerian description. The basic ideas of the lagrangian picture, and some applications, are therefore laid out in this paper.

1. Introduction

We give here the lagrangian analytical framework for the discussion of problems in fluid motion, in which we quite naturally focus attention on the motion of individual particles of fluid, just as we would do in newtonian dynamics of a system of particles. An important force field for us is the pressure gradient, which is associated with driving the motion and which therefore appears in the equation of motion for an individual particle.

In truth, of course, the existence of a pressure field represents the fact that we are dealing with an assembly of particles, and the very nature of this suggests the propriety of a field description of the flow of a vector field (velocity and pressure) at a point, the so-called eulerian description. There is a simple connection between the two descriptions, namely through the calculus: the acceleration of an individual particle is related through the chain rule to the acceleration of the assembly of particles which flow through a point.

In either description, lagrangian or eulerian, mass must be conserved in a material volume, and this yields the continuity condition, which is an important and significant equation. In the lagrangian case it can be considered as constraining the equation of motion, the balance between the particle acceleration and the pressure gradient. Thus we may regard likely candidates for the pressure field as being constrained implicitly by the continuity condition, even though the latter is explicitly representative of velocity and density.

It is incumbent upon us to refer to the important role played by vorticity, the property representative of spin or angular velocity of fluid particles. We can conceive of vorticity playing a central role since, in the eulerian description, we see the vorticity as implying the structure of the velocity field through the Biot–Savart law.

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In the lagrangian picture, on the other hand, we have an explicit formula for the vorticity in terms of the initial vorticity and the rate of strain between neighbouring fluid particles; this is Cauchy's formula, and in some circumstances it has been seen to enable the calculation of explicit solutions of considerable interest. In either formulation, we see the pressure and the vorticity, subject to the constraint of the continuity condition, as playing vital roles in the evolution of fluid motions.

We need to ask: What are the particular problems for which the lagrangian description is especially appropriate? For what problems are signal benefits to be gained from the lagrangian description? To a considerable degree answers to these questions are given by the articles in this Theme Issue of *Philosophical Transactions*.

One subject of great topicality is that of chaos associated with individual particle motions, for which one might use the description 'kinematic chaos', a term appropriate for a situation in which the dynamical (eulerian) velocity field is deterministic and non-chaotic. This subject is pursued later by H. Aref, who argues in favour of the term 'chaotic advection' in preference to an earlier usage of 'lagrangian turbulence'. A crucial point that we can make in favour of 'kinematic chaos' is that the process is not a dynamical one, but solely one of kinematics. In §3 of this paper we exploit the lagrangian frame to compare and contrast a variety of kinematical and dynamical issues and discuss the interplay (if any) between kinematical chaos and eulerian turbulence.

It would be misleading, however, to suppose that the use of a lagrangian description is concerned only with kinematic properties, or indeed exclusively with chaos. Although the relation between the lagrangian and eulerian descriptions is a kinematic one, the lagrangian form necessarily contains all the appropriate dynamical information. This feature is embodied firmly in the articles of this theme issue.

The description of the motion of 'foreign' particles in a fluid medium finds a natural place in the lagrangian formation, as does the associated concept of mixing processes and the stirring of a passive scalar contaminant. Thus M. R. Maxey's article addresses questions of particle transport and sedimentation, both chaotic and otherwise.

The idea of the straining of neighbouring fluid elements also is sensibly described in the lagrangian framework, and this is important for turbulent dispersion as has been known at least since G. I. Taylor's paper of 1921. The assumption of the 'persistence of strain' is one that has been commonly used for many years, but S. B. Pope argues later that direct numerical studies of turbulence militate against this assumption.

The Euler equations of inviscid fluid motion continue to be a subject of great mathematical speculation, not least as to whether the possibility exists for a singularity to appear in a solution within a finite time. Examples are known of this phenomenon for geometries that are semi-infinite, but questions of this type for a bounded domain or of solutions with bounded energy remain open. This topic is addressed briefly later in this article, but in more detail by H. K. Moffatt.

The Navier–Stokes problem may also have associated singular structures, especially amenable to a lagrangian description, and not unrelated to the work briefly mentioned later in this article. Unsteady boundary-layer separation of viscous flows, in which a singularity can occur in the outer regions of the boundary layer is a subject to be discussed here and is addressed by S. J. Cowley, L. Van Dommelen and S. T. Lam.

Singularities of another kind are those associated with material interfaces between two different fluids, the very nature of this problem suggesting the idea of the utilization of the lagrangian description. Some mathematical results of great interest are given by P. Constantin and L. P. Kadanoff.

Numerical studies of flows involving free surfaces (or material interfaces) form a significant area of study by the techniques of computational fluid dynamics, involving vortex elements and the lagrangian description. Progress in this field of study is described by G. R. Baker.

2. Mathematical basis

We consider a fluid that is composed of particles, a typical member of which is defined by the three parameters ϕ, ψ, χ (or by $a_i, i = 1,2,3$). These three members will be taken to represent the three coordinates of the particle at the initial time, say $t = 0$. Thus the current position $(x_j, j = 1,2,3)$ of a typical particle may be written

$$x_1 = X(a_i, t), \quad x_2 = Y(a_i, t), \quad x_3 = Z(a_i, t). \tag{2.1}$$

The equation of motion may be written

$$\rho\, \partial^2 x_j/\partial t^2 = -\partial p/\partial x_j, \tag{2.2}$$

where we consider the pressure (p) and density (ρ) to be functions of x_j and t. If, as is more often convenient in the lagrangian description, we regard p and ρ as functions of a_i rather than x_j (Lamb 1932, p. 13) we obtain

$$\rho\,(\partial^2 x_j/\partial t^2)\, \partial x_j/\partial a_i = -\partial p/\partial a_i. \tag{2.3}$$

(We shall return to a discussion of the relative merits of (2.2) and (2.3) as appropriate dynamical statements.)

The velocity of the particle is defined by

$$u_1 = \partial x_1/\partial t = X_t, \quad u_2 = \partial x_2/\partial t = Y_t, \quad u_3 = \partial x_3/\partial t = Z_t, \tag{2.4}$$

where differentiations are performed with a_i $(\equiv \phi, \psi, \chi)$ kept fixed. The vorticity

$$\omega_j(a_i, t) = \epsilon_{jkn}\, \partial u_n/\partial x_k \tag{2.5}$$

is given by Cauchy's formula (Lamb 1932, p. 205), namely

$$\rho^{-1}\, \omega_j(a_i, t) = \rho_0^{-1}\, \omega_k(a_i, 0)\, \partial x_j/\partial a_k, \tag{2.6}$$

where ρ_0 is the initial density of the particle $(t = 0)$, and $\omega_k(a_i, 0)$ is its initial vorticity. Formulae (2.4), (2.5) and (2.6) comprise the essential kinematical statements of the problem.

Finally, we have the equation of state, which we assume to be

$$p = f(\rho), \tag{2.7}$$

and the equation of continuity, namely

$$\rho J = \rho_0, \tag{2.8}$$

where

$$J = \left|\frac{\partial x_j}{\partial a_i}\right| = \frac{\partial(X, Y, Z)}{\partial(\phi, \psi, \chi)} \tag{2.9}$$

is the determinant of the matrix,

$$I = \frac{\partial x_j}{\partial a_i} \equiv \begin{bmatrix} X_\phi & X_\psi & X_\chi \\ Y_\phi & Y_\psi & Y_\chi \\ Z_\phi & Z_\psi & Z_\chi \end{bmatrix}. \tag{2.10}$$

Initial conditions require that

$$X = \phi, \quad Y = \psi, \quad Z = \chi \quad \text{at} \quad t = 0. \tag{2.11}$$

How are we to proceed with the solution of equations (2.1)–(2.11)? What are the basic principles to be used? Whether we use the eulerian or the lagrangian description, we are aware that the vorticity and the pressure play significant roles in flow development.

By utilization of (2.1), (2.4), (2.5) and (2.6) together with the continuity condition (2.8), (2.9) some detailed analysis, which extends that of Stuart (1990), can be used to show that

$$\mathscr{F}_k = 0, \quad k = 1, 2, 3, \tag{2.12}$$

where

$$\mathscr{F}_1 = Z_\chi Z_{\psi t} - Z_\psi Z_{\chi t} + Y_\chi Y_{\psi t} - Y_\psi Y_{\chi t} + X_\chi X_{\psi t} - X_\psi X_{\chi t} - Z_{\psi t0} + Y_{\chi t0}, \tag{2.13}$$

$$\mathscr{F}_2 = Z_\phi Z_{\chi t} - Z_\chi Z_{\phi t} + Y_\phi Y_{\chi t} - Y_\chi Y_{\phi t} + X_\phi X_{\chi t} - X_\chi X_{\phi t} - X_{\chi t0} + Z_{\phi t0}, \tag{2.14}$$

$$\mathscr{F}_3 = Z_\psi Z_{\chi t} - Z_\chi Z_{\psi t} + Y_\psi Y_{\chi t} - Y_\phi Y_{\psi t} + X_\psi X_{\phi t} - X_\psi X_{\phi t} - Y_{\phi t0} + X_{\psi t0}, \tag{2.15}$$

Formula (2.12) is essentially Cauchy's vorticity formula, and suffixes denote derivatives with respect to ϕ, ψ, χ, t, whereas the suffix 0 indicates the initial value of the function. Involved in the above derivation is the condition, associated with (2.11), that the initial vorticity is

$$\omega_{10} = Z_{\psi t0} - Y_{\chi t0}, \quad \omega_{20} = X_{\chi t0} - Z_{\phi t0}, \quad \omega_{30} = Y_{\phi t0} - X_{\psi t0}. \tag{2.16}$$

The scheme followed by Stuart (1988, 1990) derives particular solutions of (2.12)–(2.15) in association with the continuity condition (2.8), (2.9), with the pressure following from (2.3). In those papers, therefore, the pressure has an explicit dependence on a_i ($\equiv \phi, \psi, \chi$) and t (2.3).

For later reference we note

$$\mathscr{F}_{1t} = Z_\chi Z_{\psi tt} - Z_\psi Z_{\chi tt} + Y_\chi Y_{\psi tt} - Y_\psi Y_{\chi tt} + X_\chi X_{\psi tt} - X_\psi X_{\chi tt}, \tag{2.17}$$

with corresponding formulae for $\mathscr{F}_{2t}$ and $\mathscr{F}_{3t}$.

An alternative approach, which is based on (2.2) with p regarded as depending on x_j ($\equiv X, Y, Z$) and t, has advantages both conceptually and technically. In view of (2.7) we can define

$$\Phi(X, Y, Z, t) = \int \frac{\mathrm{d}p}{\rho} \tag{2.18}$$

and re-write (2.2) as

$$\partial^2 x_j / \partial t^2 = -\partial \Phi / \partial x_j, \tag{2.19}$$

or as

$$X_{tt} = -\Phi_X, \quad Y_{tt} = -\Phi_Y, \quad Z_{tt} = -\Phi_Z. \tag{2.20}$$

Here Φ plays the role of a potential function for the motion of a particle in three space dimensions X, Y, Z. Appropriate initial conditions are (2.11) together with

$$X_t = u_{10}(\phi, \psi, \chi), \quad Y_t = u_{20}(\phi, \psi, \chi), \quad Z_t = u_{30}(\phi, \psi, \chi) \tag{2.21}$$

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for the initial velocity field. If we substitute formulae (2.20) into (2.17) and its two companions we find

$$\mathscr{F}_{1t} = \mathscr{F}_{2t} = \mathscr{F}_{3t} = 0. \tag{2.22}$$

Thus, as derived by Stuart (1990) rather obliquely, (2.20) may be regarded as solving Cauchy's vorticity relation (2.12)–(2.15). This is the advantage in the utilization of (2.20) rather than (2.3).

It seems therefore that the apparatus of lagrangian or hamiltonian mechanics is available for the study of our problem based on (2.20). The 'energy'

$$T+\Phi = \tfrac{1}{2}(X_t^2+Y_t^2+Z_t^2)+\Phi(X,\ Y,\ Z,\ t) \tag{2.23}$$

is not in general conserved because Φ and therefore $L = T-\Phi$ may depend explicitly on t. (The 'steady' case in the eulerian sense, in which Φ depends on X, Y, Z but not on t explicitly, yields naturally Bernoulli's theorem, namely that (2.23) is constant on a streamline; in the present sense this means that (2.23) is a function of ϕ, ψ, χ and varies therefore from particle to particle in general, but is the same for particles on the same streamline.)

Implicit in the solution of (2.20) is the fact that the solution will depend in general of ϕ, ψ, χ because of (2.11), (2.21). This dependence is associated with the overwhelming constraint of continuity (2.8), (2.9). This constraint involves t only as a parameter, but derivatives with respect to the particle parameters ϕ, ψ, χ are significant, as can be seen quite easily (2.8), (2.9).

Thus, although (2.20) subject to (2.11) and (2.21) defines apparently a standard dynamical problem associated with time dependence, a lagrangian or hamiltonian problem that is, the potential function $\Phi(X, Y, Z, t)$, which is associated with pressure and density, is 'unknown'; rather, it is more accurate to say, Φ is prescribed by the relation with neighbouring particles both near and far through the continuity condition, with ρ depending implicitly on Φ through the inversion of (2.18). It is the relation with the continuity condition that renders the problem non-standard.

The work of Stuart (1990) can be re-formulated from our present point of view by setting Φ as quadratic in $x_3 = Z$. The earlier paper (Stuart 1988) has Φ quadratic in both $x_1 = X$ and $x_3 = Z$. These papers are thus concerned with structures of stagnation type. The simplicity of the present approach can be seen as follows.

If we assume that

$$\Phi = -\tfrac{1}{2}A^2X^2-\tfrac{1}{2}C^2Z^2+\Psi(Y,t), \tag{2.24}$$

where A and C can be functions of t, then

$$X_{tt}-A^2X = 0, \tag{2.25}$$

$$Z_{tt}-C^2Z = 0, \tag{2.26}$$

$$Y_{tt}+\Psi_Y = 0. \tag{2.27}$$

With $A \equiv 0$ as a special case, and applying conditions (2.11), (2.21), we find

$$X = \phi(1+t\ u_0(\psi)) \tag{2.28}$$

$$Z = \chi(\cosh Ct+w_0(\psi)\sinh Ct). \tag{2.29}$$

A simple calculation shows that (2.12)–(2.15) are satisfied, as are formulae (2.22), provided Y is a function of ψ and t only.

For the incompressible case ($\rho = \rho_0$) (2.8) and (2.9) yield a relation between Y and ψ by a quadrature

$$[1+tu_0(\psi)]\,Y_\psi[\cosh Ct+w_0(\psi)\sinh Ct] = 1. \tag{2.30}$$

With $u_0(\psi)$ and $w_0(\psi)$ given as initial data, (2.30) can be evaluated as in Stuart (1988, 1990) and Ψ follows from (2.27).

The compressible case (2.7) and other aspects of the problem defined by (2.20), (2.8), (2.9), (2.11), (2.21) will be discussed elsewhere. For the present we emphasize the simple dynamical nature of problem (2.20), but subject to the formidable constraint (2.8), (2.9).

3. Kinematical and dynamical issues

As we have discussed earlier, one currently popular aspect of the lagrangian picture is kinematic chaos. The appeal is that, since very simple (deterministic) velocity fields can generate chaotic particle trajectories, many of the concepts and tools of dynamical systems theory can be used and hence play a useful role in describing real physical problems. However, little is understood about the role or significance of kinematic chaos in régimes where the velocity fields themselves start to fluctuate chaotically on all (or many) temporal and spatial scales. Thus the effect of eulerian 'turbulence' on kinematic chaos and the converse issue of the influence of kinematic chaos on eulerian turbulence appears to be a completely open question. Arnold (1965) has suggested that in some circumstances chaotic streamlines may facilitate the onset of eulerian turbulence. On the other hand, as will be discussed later, dynamical systems intuition suggests that the latter may suppress the former. Some of the dichotomy between kinematical and statistical considerations is illustrated by considering the fundamental process of (nearby) particle separation. That nearby particles should separate on average in a turbulent flow field has been a long-held assumption; although a rigorous demonstration of this was given only relatively recently by Cooke (1969, 1971) who showed that for isotropic turbulence (or locally isotropic turbulence with a restriction on timescales (Monin & Yaglom 1975)) the mean separation grows exponentially, namely $\ln|l(\tau)|/l(0) \geqslant 0$ for all $\tau > 0$, but that it saturates in the limit $\tau \rightarrow \infty$. It is important to consider the role of different (eulerian) scales in the separation process. Separation is determined by the local velocity gradients, the largest of which are associated with the smallest scales. Two infinitesimally close particles will only separate significantly over regions in which the straining field is well correlated, namely the Kolmogorov microscale in fully developed turbulence. For particle separations larger than this, say in the inertial range, this correlation is lost. (Indeed, little is known about the statistics of gradient quantities in the lagrangian picture.) In this régime one can argue that at a separation l, where $l_k \ll l \ll l_0$ and l_k and l_0 denote the micro- and integral scales respectively, the particles are primarily pushed further apart by the eddies of size l. These larger eddies contain more energy with the result that as the process progresses the separation becomes faster than the ballistic rate. A standard dimensional analysis yields the famous Richardson law $\langle l^2 \rangle^{\frac{1}{2}} \approx \epsilon^{\frac{1}{6}} \tau^{\frac{3}{2}}$ where ϵ is the universal rate of energy dissipation. The (presumed) loss of straining field correlation beyond the smallest scales is an example of where a dynamical systems point of view of particle separation, as embodied in the calculation of Lyapunov exponents, is different from the statistical one; in the former context a precise correlation of the velocity gradients and the orientation of the separation vector is assumed along the entire system trajectory (particle path). If this correlation is lost these exponents (if they can still be defined) might reasonably be expected to lessen; hence the intuition that eulerian turbulence might suppress the extent of kinematic chaos.

Even at the smallest scales the actual rate of local (exponential) separation is far

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from obvious. For many years it was assumed that, at the smallest scales, the straining field is 'persistent', that is the directions of the principal rates of strain vary more slowly than the strain rates themselves. The consequence of this assumption is that an infinitesimal line element has time to align itself with the largest strain rate and be significantly stretched. Such a model leads to a local separation rate proportional to the largest (mean) eigenvalue of the rate-of-strain tensor (Batchelor & Townsend 1956). As will be discussed in the paper by S. B. Pope, numerical evidence suggests that this persistence assumption is probably incorrect with the consequence that the separation rate is reduced. Even so the precise kinematics of alignment – a variety of effects involving the vorticity and the rotation rates of the strain axes competing to move the line element in and out of alignment – is still not well understood.

At the dynamical level, the comparison of scaling laws for lagrangian and eulerian quantities raises other important issues. An illustration is provided by comparing the relative behaviour of the frequency spectra of the velocity autocorrelation in the lagrangian and eulerian frames. Here we consider the one-point two-time eulerian quantity

$$C_{ij}^{(\mathrm{E})}(\tau) = \langle u_i(x,t)\, u_j(x,t+\tau)\rangle \tag{3.1}$$

and the corresponding lagrangian quantity

$$C_{ij}^{(\mathrm{L})}(\tau) = \langle u_i(a,t)\, u_j(a,t+\tau)\rangle, \tag{3.2}$$

where a denotes a fluid particle label. In the theory of isotropic, homogeneous turbulence the study of time correlations, as opposed to space correlations, has met with only modest success. If one assumes a Kolmogorov cascade with the lowest frequencies determined by the integral scales, i.e. $\omega_0 = u_0/l_0$, and the cut-off frequency determined by the micro-scales, i.e. $\omega_k = u_k/l_k$, it follows from simple scaling arguments (if they exist for time correlations) that the frequency spectrum (i.e. the Fourier transform of $C(\tau)$) of either correlation behaves as

$$\phi(\omega) = \epsilon\omega^{-2}, \tag{3.3}$$

where ϵ is the (universal rate) of energy dissipation. In the eulerian frame, however, one should allow for the possibility that the large-scale, energy containing, eddies advect the smaller eddies (this is sometimes referred to as the 'random Taylor' or 'sweeping' hypothesis). At an intuitive level this effect can be estimated by observing that the mean square advective term $\langle(v\cdot\nabla v)^2\rangle$ can be decomposed into $\langle v^2\rangle\,\langle(\nabla v)^2\rangle$ if the large and small scales become uncorrelated. A consequence of this is that the highest frequency is effectively shifted to $\omega = u_k/l_0$ and the spectrum becomes (Tennekes 1975)

$$\phi(\omega) = \epsilon^{\frac{2}{3}} u_0^{\frac{2}{3}}\, \omega^{-\frac{5}{3}}, \tag{3.4}$$

which is non-universal since it now depends on some typical large-scale velocity u_0. It seems reasonable to suggest that the sweeping result, $\omega^{\frac{5}{3}}$, corresponds to the eulerian spectrum and the non-sweeping result, ω^{-2}, corresponds to the lagrangian spectrum since, by definition, there cannot be sweeping in the lagrangian frame. One consequence of this is that the ratio of some lagrangian (micro) timescale to the corresponding eulerian timescale scales as $Re^{\frac{1}{4}}$. This would imply that along a lagrangian orbit the velocity of a particle becomes ever better correlated relative to its eulerian counterpart as a function of increasing Reynolds number. There is, as yet, little evidence to verify this contention. (Some interesting numerical tests are reviewed here by S. B. Pope.)

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The notion of sweeping is not without controversy. Results of Yakhot *et al.* (1988), using newly developed renormalization group (RNG) methods, suggests that there is, in fact, no sweeping. In other words the large- and small-scale motions are not decorrelated, with the result that the eulerian spectrum scales more like ω^{-2}. Chen & Kraichnan (1989) have argued that such a result is not unexpected since correlations between large and small scales are, in effect, built into the RNG approach. However, a recent investigation by Nelkin & Tabor (1990), using simpler statistical ideas, reinforces the validity of the sweeping hypothesis. Here the idea is to identify the crucial advective term with the gradient of the Reynolds stress tensor, i.e. $(v\cdot\nabla)v = \partial(v_\alpha v_\beta)/\partial x_\beta$. The fluctuation spectrum of the diagonal part is just the kinetic energy spectrum $E_K(k)$, where $K = v^2$ is the local kinetic energy spectrum per unit mass. Standard Kolmogorov scaling arguments suggest that E_K scales as $k^{-\frac{7}{3}}$. However, it is easy to show that in this case the contribution to the mean square acceleration (and hence the frequency spectrum) is consistent with the non-sweeping result (3.3). By contrast, it is also easy to show that if E_K scales as $k^{-\frac{5}{3}}$ one is led to the sweeping result (3.4). Gratifyingly there is strong experimental evidence supporting the latter form of E_K (Van Atta & Wyngaard 1975). Although this supports the sweeping hypothesis the above arguments ignore the scaling properties of the off-diagonal elements of the Reynolds stress. Their precise role is open to debate. One possibility is that, in view of the standard identity $(v\cdot\nabla v) = \frac{1}{2}\nabla(v^2) - v\times\omega$, the off-diagonal terms might themselves be subject to the decorrelation assumption, namely $\langle\nabla\times\omega\rangle = \langle v^2\rangle\langle\omega^2\rangle$, and hence make a similar contribution to the mean square acceleration as the diagonal terms. Either way the above discussion should illustrate the need for much further work on the relative behaviours of eulerian and lagrangian quantities.

A quantity, that captures both kinematical and dynamical information, is the second invariant of the velocity gradient tensor, namely

$$\sigma^2 = \tfrac{1}{2}\operatorname{tr}A^2 \tag{3.5}$$

where $(A)_{ij} = \partial u_i/\partial x_j$. This quantity gives a simple and compact measure of the straining and rotational components of the motion since it is easily shown that

$$\sigma^2 = \tfrac{1}{2}\sum_i (s_i^2 - \omega_i^2), \tag{3.6}$$

where s_i and ω_i are the eigenvalues of the rate-of-strain and vorticity tensors respectively (i.e. the real and imaginary parts of the eigenvalues of A). In two dimensions σ^2 is precisely the gaussian curvature of the stream function. Clearly if $\sigma^2 < 0$ the motion is rotation dominated, whereas for $\sigma^2 > 0$ the motion is strain dominated. The behaviour of σ^2 in the lagrangian frame is of considerable interest. For a given initial particle position, a, the quantity $\sigma^2 = \sigma^2(a, t)$ characterizes the 'strain history' of the particle, which is of value in understanding the dynamics of small deformable bodies, such as polymers, in flow fields (Dresselhaus & Tabor 1989). By taking the time average of the real and imaginary parts of σ itself (i.e. $\sigma = (\operatorname{tr}A)^{\frac{1}{2}}$) along a lagrangian orbit the ratio

$$\chi = \sigma_{\mathrm{Re}}/\sigma_{\mathrm{Im}} \tag{3.7}$$

gives a simple measure of 'stretch–fold' ratio of particle orbits, i.e. the ratio of overall strain dominated to vorticity dominated motion. Such a quantity is relevant to quantifying mixing efficiency of passive scalars. The quantities σ^2 and χ capture

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valuable kinematic information, which is distinct from that contained in the traditional Lyapunov exponents which are, in effect, the long time average of the real parts of the eigenvalues of A (A is the 'tangent map' in the language of dynamical systems) and hence contain no 'rotational' information. At the same time, σ^2 is of considerable fluid dynamical interest since, as is well known (in the eulerian context)

$$\sigma^2 = \nabla \cdot (u \cdot \nabla u) = -\nabla^2 p, \tag{3.8}$$

where p is the pressure field (here we are setting the density, ρ, to unity). In many numerical simulations this Poisson equation is used to compute the velocity field from a given pressure field. Thus a study of the autocorrelation in the lagrangian frame, namely

$$P(\tau) = \langle \sigma^2(a,\, t)\, \sigma^2(a,\, t+\tau) \rangle, \tag{3.9}$$

can cast light on both the nature of pressure fluctuations and the kinematic details of particle orbits.

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